RESEARCH STATEMENT

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My primary research area is algebraic geometry, tropical mathematics, valuation theory, and Berkovich spaces. More specifically, I am interested in the homotopy type of Berkovich spaces over trivially valued fields. I use techniques from birational geometry and cubical hyperresolutions to study the homotopy type of Berkovich spaces. I am also interested in algebraic structures of tropical geometry and its interaction with Berkovich spaces.

1. Tropical Geometry

1.1. **Overview.** Tropical geometry, that can be viewed as combinatorial shadow of algebraic geometry, has been received a lot of attention in recent years. One can broadly described tropical geometry as algebraic geometry over the semifield $\mathbb{T} := (\mathbb{R} \cup \{\infty\}, \min, +)$ of tropical numbers where minimum is the tropical addition and addition is the tropical multiplication. Under these operations, a tropical polynomial $f \in \mathbb{T}[x] = \mathbb{T}[x_1, x_2, ..., x_k]$ is just a piecewise linear function $f\colon \mathbb{T}^k \to T$ and it is convenient to define tropical hypersurface $V(f)$ as nonlinear locus of this function. If I is an ideal of $\mathbb{T}[x]$, then the associated tropical variety $V(I) = \bigcap_{f \in I} V(f)$ is a polyhedral complex. See [MS15].

If one follows Zariski's approach in classical algebraic geometry to define topology on tropical varieties, then there would be an inconsistency: the induced topology on tropical variety $V \subseteq \mathbb{T}^k$ is not an intrinsic invariant of V *i.e.* it depends on the embedding of V into affine space $\mathbb{T}^k$. See Example 17 in [Yag16].

Researchers addressed this issue in several different approaches. In [Mik06], [MS15], and [Pay09] they simply equip the affine space $\mathbb{T}^k$ with the Euclidean topology. The authors in [BE13] and [JM15] work with congruences instead of ideals. One could easily verify that the topology on $\mathbb{T}^k$ whose closed sets are congruence varieties is the Euclidean topology (see Theorem 18 and Proposition 29 in [Yag16]). Jeffrey and Noah Giansiracusa [GG16] define *strong Zariski topology* by using the $\mathbb{T}$-points of arbitrary subschemes as the closed sets and they show it gives the Euclidean topology on $\mathbb{T}^k$. In [IR10] and [Izh09] Izhakian and Rowen extend $\mathbb{T}$ to *supertropical semiring* which is not an idempotent semiring. In addition, their topology is more complicated than the Euclidean topology. See [Izh08] Section 1.

1.2. **Tropical dual numbers.** In [Yag16], we addressed this issue by extending $\mathbb{T}$ to *tropical dual numbers* $\widetilde{\mathbb{T}}$ by adjoining a nonzero nilpotent element ϵ to $\mathbb{T}$ such that $\epsilon^2 = 1_{\mathbb{T}}$. In other words, a tropical dual number can be written, in a unique way, as $a + b\epsilon$ where $a, b \in \mathbb{T}$ and $\epsilon^2 = 1_{\mathbb{T}}$. By this innovation, one can work with honest ideals, instead of congruences, and recover the Euclidean topology on $\mathbb{T}^k$ in Zariski-type approach. More precisely, we define *tropical Zariski topology* on $\mathbb{T}^k$ by using closed sets $V(\mathscr{I})$ where $\mathscr{I}$ is an ideal in $\widetilde{\mathbb{T}}[x]$.

Theorem 1. *The tropical Zariski topology on $\mathbb{T}^k$ is equal to the Euclidean topology.*

There is a tight connections between tropical dual numbers and congruences as follows. Roughly speaking, a congruence E of $\mathbb{T}[x]$ is an equivalence relation on $\mathbb{T}[x]$ which respects the semiring operations. Note that in semiring $\mathbb{T}[x]$ there is no subtraction and the idea of working with congruences is that the pair $(f,g) \in E$ represents the subtraction $f-g$. On the other hand, in $\widetilde{\mathbb{T}}[x]$ somehow we add "negative $1_{\mathbb{T}}$", which we represent with a new symbol ϵ, to $\mathbb{T}[x]$ by taking the quotient of $\mathbb{T}[x][\epsilon]$ by congruence $\langle \epsilon^2 \sim 1_{\mathbb{T}} \rangle$. Our idea is to represent a pair (f,g), *i.e.* the imaginary subtraction $f-g$, by $f + g\epsilon \in \widetilde{\mathbb{T}}[x]$. Moreover, since in a ring one has $(f-g)(f'-g') = (ff'+gg') - (fg'+f'g)$, the authors in [BE13] consider the twisted product of pairs as $(f,g) \times (f',g') = (ff'+gg', fg'+f'g)$. This is compatible with the multiplication of $\widetilde{\mathbb{T}}[x]$ because $(f+g\epsilon)(f'+g'\epsilon) = (ff'+gg') + (fg'+f'g)\epsilon$. Finally, for congruence variety $V(E)$ we constructed an ideal $\mathscr{I}$ in $\widetilde{\mathbb{T}}[x]$ such that $V(E) = V(\mathscr{I})$.

1.3. **Future research projects.** Here I will explain briefly a few projects that I have in mind for the future.

The next project in this direction would be to find a satisfactory definition for prime ideal in $\widetilde{\mathbb{T}}[x]$ and formulate tropical Nullstellensatz. Under the definition in [BE13], that proper congruence E of $\mathbb{T}[x]$ is prime if $\mathbb{T}[x]/E$ is an integral domain, one can build an infinite sequence of prime congruences $E_1 \subset E_2 \dots \subset E_n \subset E_{n+1} \dots$ in semiring $\mathbb{T}[x]$ but we expect that the length of the longest chain of prime congruences be k. Also according to [JM15], proper congruence E of $\mathbb{T}[x]$ is prime if $(f,g) \times (f',g') \in E$ implies $(f,g) \in E$ or $(f',g') \in E$. We think that this definition is too strong because one can show congruence E is prime only if $V(E)$ is empty or a single point.

Maclagan and Rincon, inspired by [GG16], in their recent paper [MR16], introduced *tropical ideals* of $\mathbb{T}[x]$ as those ideals which are, when regarded as $\mathbb{T}$-submodules of the free module $\mathbb{T}[x]$, tropical linear spaces. It appears that the bend loci of these special ideals are balanced polyhedral complexes that people in tropical geometry normally work with, whereas the bend loci of arbitrary ideals are far less restricted. It would be interesting to think about the topology on $\mathbb{T}^k$ whose closed sets are bend loci of only tropical ideals in $\widetilde{\mathbb{T}}[x]$.

2. Berkovich Spaces

2.1. **Overview.** Over the field of complex numbers, there is a close relation between algebraic geometry and analytic geometry. See Serre's GAGA [Ser55]. There have been many attempts by mathematicians to build a suitable analytic theory over non-archimedean fields *e.g* classical analytic theory, Tate's rigid analytic geometry, Raynaud's theory of formal models, etc. In late 80's, Berkovich discovered a satisfactory version of analytic geometry over non-archimedean fields and established its connections with algebraic geometry [Ber90].

Let k be a field which is complete with respect to non-archimedean norm $\|\cdot\|$[1] and let X be a variety over k. The Berkovich analytic space X^{an} is the set of all pairs $(k_p, |\cdot|)$ where k_p is residue field of some point $p \in X$ and $|\cdot|$ is an extension of $\|\cdot\|$ to k_p. There is an alternative description as follows. One covers X by affine open charts $U_i = \mathrm{Spec} A_i$ and then U_i^{an} is the set of all multiplicative seminorms on A_i whose restriction to k is $\|\cdot\|$; then they glue U_i^{an} to obtain X^{an}.

The topology on X^{an} is the weakest topology such that for each affine open $U = \mathrm{Spec} A$ of X and each $f \in A$, U^{an} is open in X^{an} and the map $\varphi_f \colon U^{\mathrm{an}} \to [0, +\infty)$ given by $|\cdot| \mapsto |f|$ is continuous. The analytic spaces X^{an} has the structure sheaf of convergent power series,

[1] For instance, $\mathbb{Q}_p$ with p-adic norm, Laurent series $\mathbb{C}((t))$ with t-adic norm, or $\mathbb{C}$ with trivial norm.

see Chapter 2 in [Ber90], however, we are interested in the homotopy type of the underlying topological space of X^{an}.

Berkovich spaces are related to tropical varieties. One important result is the theorem of Payne [Pay09]: for affine variety X the tropicalizations $\mathbf{Trop}(X,i)$ with respect to embeddings into affine spaces $i : X \hookrightarrow \mathbb{A}^m_k$ form an inverse system and there is a natural homeomorphism $X^{\mathrm{an}} \cong \varprojlim_i \mathbf{Trop}(X,i)$. In [GG16] they show that one can obtain the Berkovich space X^{an} by a single tropicalization with respect to embedding into an infinite dimensional affine space $X \hookrightarrow \widehat{X}$ which is called the universal tropicalization of X, *i.e.*, $X^{\mathrm{an}} \cong \mathbf{Trop}_{\mathrm{univ}}(X)$.

2.2. **The homotopy type of X^{an}.** One general approach to study the homotopy type of analytic space X^{an}, as stated in [Ber90], is to construct a deformation retraction of X^{an} in the following way: one finds an action of an analytic group G^{an} on X^{an} and a continuous family of analytic subgroups $\{G^{\mathrm{an}}_t\}_{t\in[0,1]}$ such that $G^{\mathrm{an}}_0 = \{e\}$, $G^{\mathrm{an}}_1 = G^{\mathrm{an}}$, and $G^{\mathrm{an}}_{t_1} \subseteq G^{\mathrm{an}}_{t_2}$ if $t_1 \leq t_2$. Each G^{an}_t has a special maximal point g_t called the *Shilov point* of G^{an}_t. Then, $X^{\mathrm{an}} \times [0,1] \to X^{\mathrm{an}} : (x,t) \mapsto g_t * x$ is a deformation retraction of X^{an}.

From now on, let X be a variety over $\mathbb{C}$ where $\mathbb{C}$ is equipped with the trivial norm *i.e.* $\|0\| = 0$ and $\|a\| = 1$ for $a \in \mathbb{C}^*$. If X is a smooth projective variety and E a simple normal crossing divisor in X, then, following the ideas of Berkovich [Ber90], [Ber99]; Thuillier [Thu07] constructs a deformation retraction of X^{an} onto a small polyhedral complex $\mathrm{Sk}(X,E)$ inside X^{an} which is called the *skeleton* of X^{an} and is build from the dual complex of E. In particular, this implies that X^{an} is contractible when X is smooth.

For arbitrary projective variety, it is proved in [Ber09], that the singular cohomology of the underlying topological space of X^{an} is isomorphic to weight zero cohomology of X.

$$H^q(X^{\mathrm{an}},\mathbb{Q}) \simeq W_0 H^q(X,\mathbb{Q})$$

Now assume X has an isolated singularity and let $f : Y \to X$ be a log resolution with exceptional divisor E. We have the following result

Theorem 2 (T. de Fernex, L. Fantini, K. Yaghmayi)**.** *If X has an isolated singularity, then X^{an} is homotopy equivalent to the suspension of the dual complex of E.*

Using the results of [dFKX14], we immediately obtain

Corollary 3. *If X has an isolated kawamata log terminal (or klt for short) singularity, then X^{an} is contractible.*

The brief idea of the proof of the Theorem 2 is as follows: We show deformation retraction $H_Y : Y^{\mathrm{an}} \times [0,1] \to Y^{\mathrm{an}}$, which exists by [Thu07], descends to a continuous map $H_X : X^{\mathrm{an}} \times [0,1] \to X^{\mathrm{an}}$ such that the following diagram commutes

$$\begin{tikzcd} Y^{\mathrm{an}} \times [0,1] \arrow[r, "H_Y"] \arrow[d, "f^{\mathrm{an}}\times id"'] & Y^{\mathrm{an}} \arrow[d, "f^{\mathrm{an}}"] \\ X^{\mathrm{an}} \times [0,1] \arrow[r, "H_X"] & X^{\mathrm{an}} \end{tikzcd} \tag{1}$$

The skeleton $\mathrm{Sk}(Y,E)$ is the natural compactification of the cone over the dual complex of E. In the process of descending, the boundary of $\mathrm{Sk}(Y,E)$, sometimes called the part at infinity, maps into a single point, and therefore, we obtain the suspension of the dual complex of E.

2.2.1. *Connections with cubical hyperresolutions.* The geometric realization $|E_\bullet|$ of E is a CW complex which encodes (1) the incidence properties of irreducible components of E and (2) the topological properties of each irreducible component of E. If π_0 be the connected component functor which assigns the set of connected components to a topological space, then $|\pi_0(E_\bullet)|$ only encodes the combinatorics of intersections of irreducible components of E and one can verify that it has the homotopy type of the dual complex of E. On the other hand, each irreducible component E_i of E is smooth, and therefore, E_i^{an} is contractible. Because taking connected component and contacting to a point have the same effect on irreducible components of E, one can show that $|E_\bullet^{\mathrm{an}}|$ and $|\pi_0(E_\bullet)|$ are homotopy equivalent. Therefore, $|E_\bullet^{\mathrm{an}}|$ has the homotopy type of the dual complex of E.

Following the steps in [PS08], one can construct a cubical hyperresolution of size $\dim(X)+1$ for X, however, it would not be immediate to see the relation between the geometric realizations $|\pi_0(X_\bullet)|$ and $|\pi_0(E_\bullet)|$. We construct a cubical hyperresolution for E whose size is the number of irreducible components in E and then we use it to obtain a cubical hyperresolution for X. By this approach we are able to show that $|\pi_0(X_\bullet)|$ is homotopy equivalent to the suspension of $|\pi_0(E_\bullet)|$. Our construction also sheds some light on the CW structure of $f^{\mathrm{an}}(\mathrm{Sk}(Y,E))$, *i.e.*, the skeleton of X.

2.3. **Ongoing research project.** Our goal is to extend Corollary 3 to arbitrary *klt* singularities. Note that outside of the singular locus of X, the map f^{an} is isomorphism

$$f^{\mathrm{an}}\colon Y^{\mathrm{an}} \setminus E^{\mathrm{an}} \xrightarrow{\simeq} X^{\mathrm{an}} \setminus (X_{\mathrm{sing}})^{\mathrm{an}}$$

therefore, the homotopy H_Y descends, in a unique way, to a continuous map such that diagram 1 commutes. The difficult part is to show that over E^{an} the homotopy H_Y respects the fibers of f^{an}, that is, if $y_1, y_2 \in E^{\mathrm{an}}$ and $f^{\mathrm{an}}(y_1) = f^{\mathrm{an}}(y_1)$, then $f^{\mathrm{an}}(H_Y(y_1,t)) = f^{\mathrm{an}}(H_Y(y_1,t))$ for every $t \in [0,1]$.

In the case of isolated singularity this holds for any log resolution $f\colon Y \to X$ because the whole space E^{an} is the fiber of f^{an} over a single point and homotopy H_Y stabilizes E^{an}, see [Thu07] and [ACP15]. For an arbitrary *klt* singularity, we want $H_Y\colon Y^{\mathrm{an}} \times [0,1] \to Y^{\mathrm{an}}$ to stabilize the fibers of f^{an} over $(X_{\mathrm{sing}})^{\mathrm{an}}$ and it is not clear if it is the case for arbitrary log resolution. Our idea is to choose a special log resolution $f\colon Y \to X$ and then make some modifications by étale maps, using the results in [Thu07] that the definition of the homotopy H_Y is independent of the choice of étale cover.

Cubical hyperresolution and their geometric realizations provide an alternative approach to study the homotopy type of Berkovich spaces: since X and our geometric realization $|X_\bullet|$ are metric spaces, if the augmented geometric realization $|\epsilon|\colon |X_\bullet| \to X$ has contractible fibers, then by applying the main theorem in [Sma57], we can show $|\epsilon|$ is homotopy equivalence. However, after applying the analytification functor, it is not clear whether $|\epsilon^{\mathrm{an}}|\colon |X_\bullet^{\mathrm{an}}| \to X^{\mathrm{an}}$ is a homotopy equivalence, because the Berkovich analytic spaces are not metrizable in general. With some work, we are able to replace $|X_\bullet^{\mathrm{an}}|$ with a homotopy equivalent metric space. If we could resolve the metrizability issue for X^{an}, then we can show X^{an} has the homotopy type of the suspension of the dual complex of E.

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